A Measure of Estimated Accuracy Using Singular Values for TOA Location System

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Abstract In a TOA (Time of Arrival) system such as the GPS, a target location can be estimated from several range (distance) measurements between a target location and a reference point at known location. However, accuracy of estimated target location is very sensitive to target and reference points geometry. In the present TOA system, the DOP（Dilution of Precision） is used as a measure for unique solution and the estimated location accuracy. However, the DOP is calculated only from the diagonal elements of the 4×4 matrix $A^T A$ (A^T is the transpose of A) and it is assumed that $A^T A$ is invertible and variances of a range measurement error are the same value. Here, the $n\times 4$ matrix A can be calculated from n reference points and the nominal value of the target's location. In this paper, we provide a measure of estimated target location accuracy for a TOA system. This measure is the minimum singular value $\lambda_{\min}(A)$ of a matrix A. We show that we could obtain good estimation when $\lambda_{\min}(A)$ is large.

Keyword TOA, GPS, range measurement, singular value, error analysis, estimated accuracy

1. Introduction

This paper deals with a range-based target location algorithm. In a TOA (Time of Arrival) system such as the GPS (Global Positioning System), a target location can be estimated from several range (distance) measurements between a target location and a reference point at known location as shown in Fig.1 [1]~[3].

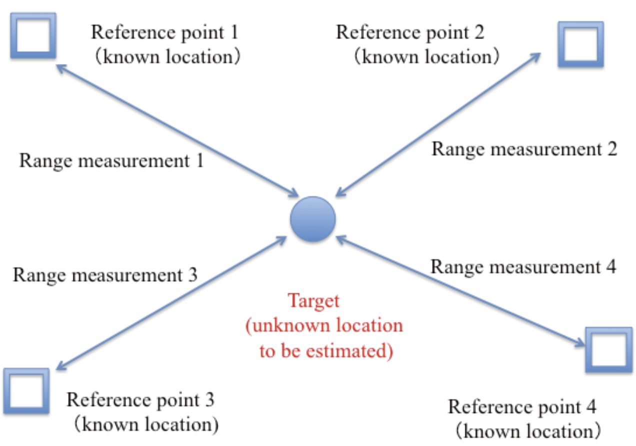

Fig.1 TOA

There are two approaches to obtain these range measurements. The first approach is to obtain the range measurements between a receiver at known location and the target that have a transmitter. The second approach is to obtain the range measurements between a transmitter at known location and the target that have a receiver such as GPS [1],[2].

In this paper, we assume that the clocks of all reference points are synchronized and there exists a clock offset between the transmitter clock and the receiver clock in a TOA system. Then, it is necessary to obtain at least four range measurements to estimate 3-dimensional target location and the clock offset. Actually, we can estimate these four unknowns using the weighed least-squares method [1]~[3].

However, accuracy of estimated target location is very sensitive to target and reference points geometry.

In the present TOA system, the DOP（Dilution of Precision） is used as a measure for unique solution and the estimated location accuracy. However, the DOP is calculated only from the diagonal elements of the 4×4 matrix $A^T A$ (A^T is the transpose of A) and it is assumed that $A^T A$ is invertible and variances of a range measurement error are the same value. Here, the $n\times 4$ matrix A can be calculated from n reference points and the nominal value of the target's location.

In this paper, we provide a measure of estimated target location accuracy for a TOA system without these assumptions. This measure is the minimum singular value $\lambda_{\min}(A)$ of the matrix A. We also illustrate the analytical results of the estimated location accuracy using this singular value.

2. Location system

In this chapter, we assume that there exist n reference points. Let A^T be the transpose of a matrix A and I_n be the $n \times n$ unit matrix.

2.1. Range measurement model

Let $\underline{B}_i$ be a known location vector of an $i(i=1,2,\cdots,n)$ th reference point in 3-dimensional Cartesian coordinates given by:

$$\underline{B}_i=(x_i,y_i,z_i)^T \tag{1}$$

Let $\underline{L}$ be an unknown location vector of the target in 3-dimensional Cartesian coordinates given by:

$$\underline{L}=(x,y,z)^T \tag{2}$$

Then, the true range (distance) R_i between the target location and the ith reference point is defined as:

$$R_i=f_i(x,y,z) \tag{3}$$

where

$$f_i(x,y,z)=\sqrt{(x_i-x)^2+(y_i-y)^2+(z_i-z)^2} \tag{4}$$

Therefore, the range measurement R_{io} between a target location and the i th reference point can be obtained as:

$$R_{io}=R_i+S+v_i \tag{5}$$

where R_i+S is the noiseless pseudorange, S is the range offset caused by the clock offset between the transmitter clock and the receiver clock, v_i is the random range measurement noise.

Then, we obtain the following property using the total differential.

(Property1) Let x_0,y_0,z_0 be an initial (nominal) estimate of the target location. Then the following result holds:

$$\Delta R_{io}=\begin{pmatrix} \alpha_i & \beta_i & \gamma_i & 1 \end{pmatrix}\underline{a}+v_i \tag{6}$$

where

$$\Delta R_{io}=R_{io}-f_i(x_0,y_0,z_0) \tag{7}$$

$$\underline{a}=(x-x_0,y-y_0,z-z_0,S)^T \tag{8}$$

$$\alpha_i=-\frac{x_i-x_0}{f_i(x_0,y_0,z_0)},\beta_i=-\frac{y_i-y_0}{f_i(x_0,y_0,z_0)},\gamma_i=-\frac{z_i-z_0}{f_i(x_0,y_0,z_0)} \tag{9}$$

2.2. Linear model

The following linear model can be obtained from Eq.(6) when i (out of n) reference points are used:

$$\underline{b}(i)=A(i)\underline{a}+\underline{v}(i) \tag{10}$$

where

$$\underline{b}(i)=(\Delta R_{1o},\Delta R_{2o},\cdots,\Delta R_{io})^T \tag{11}$$

$$A(i)=\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & 1 \\ \alpha_2 & \beta_2 & \gamma_2 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_i & \beta_i & \gamma_i & 1 \end{pmatrix} \tag{12}$$

$$\underline{v}(i)=(v_1,v_2,\cdots,v_i)^T \tag{13}$$

and we call $A(i)$ a placement matrix. By the way, we assume that range measurement noises are zero mean and uncorrelated:

$$E[\underline{v}(i)]=\underline{0} \tag{14}$$

$$V(i)=E[\underline{v}(i)\underline{v}^T(i)]=diag\{\ \sigma_1^2,\ \ \sigma_2^2,\ \ \cdots\ \ ,\sigma_i^2\ \} \tag{15}$$

Here, $E[\ ]$ indicates the mean, and $diag\{a_1,a_2,\cdots,a_i\}$ indicates the diagonal matrix whose elements are $a_1,a_2,\cdots,a_i$.

2.3. Weighed least-squares method

The following property shows that we can estimate the target location using the weighed least-squares method[4],[5]. Namely, we minimize the following equation to obtain an optimal estimate $\underline{\hat{a}}(i)$ for the location vector and the range offset:

$$J(i)=(\underline{b}(i)-A(i)\underline{\hat{a}}(i))^T V(i)^{-1}(\underline{b}(i)-A(i)\underline{\hat{a}}(i)) \tag{16}$$

Then, we obtain the following property.

(Property2) Let $A(i)^T V(i)^{-1}A(i)$ be invertible. Then the following result holds:

$$\underline{\hat{a}}(i)=\left(A(i)^T V(i)^{-1} A(i)\right)^{-1} A(i)^T V(i)^{-1} \underline{b}(i) \quad (17)$$

The following property shows that $\underline{\hat{a}}(i)$ is the unbiased estimate and its estimated covariance matrix.

(Property3) The following result holds:

$$E\left[\underline{\hat{a}}(i)\right]=\underline{a}(i) \quad (18)$$

$$E\left[\left(\underline{\hat{a}}(i)-\underline{a}(i)\right)\left(\underline{\hat{a}}(i)-\underline{a}(i)\right)^T\right]=\left(A(i)^T V(i)^{-1} A(i)\right)^{-1} \quad (19)$$

3. DOP[1],[2]

In this section, we summarize DOP (Dilution of precision) that is widely used to measure the estimated location error in TOA.

3.1. Definition

Let the range measurement noises have approximately equal variance σ^2. Then, from Eq.(15), we can obtain the following equation:

$$V(i)=\sigma^2 I_i \quad (20)$$

As a result, Eq.(19) can be written as the following equation:

$$E\left[\left(\underline{\hat{a}}(i)-\underline{a}(i)\right)\left(\underline{\hat{a}}(i)-\underline{a}(i)\right)^T\right]=\sigma^2\left(A(i)^T A(i)\right)^{-1} \quad (21)$$

Furthermore, if we omit the off-diagonal elements of $A(i)^T A(i)$, we can obtain the following equation:

$$\left(A(i)^T A(i)\right)^{-1}=diag\left\{\ \sigma_x^2,\ \sigma_y^2,\ \sigma_z^2\ ,\sigma_t^2\ \right\} \quad (22)$$

where σ_x is an x DOP, σ_y is a y DOP, σ_z is a z DOP and σ_t is a t DOP. Then, we can define the following DOPs:

$$\mathrm{GDOP}=\sqrt{\sigma_x^2+\sigma_y^2+\sigma_z^2+\sigma_t^2} \quad (23)$$

$$\mathrm{PDOP}=\sqrt{\sigma_x^2+\sigma_y^2+\sigma_z^2} \quad (24)$$

$$\mathrm{HDOP}=\sqrt{\sigma_x^2+\sigma_y^2} \quad (25)$$

$$\mathrm{VDOP}=\sigma_z \quad (26)$$

3.2. Improvement points

We can compute various DOPs only using diagonal elements of $A(i)^T A(i)$ assuming that $A(i)^T A(i)$ is invertible. We also assume the variances of a range measurement error are the same value. Therefore, we propose a new measure of estimated target location accuracy without these assumptions in order to investigate the estimated accuracy more exactly.

4. A measure of estimated accuracy using singular values

4.1. Problem formulation

We should check $\underline{\hat{a}}(i)$ $(i=4,5,\cdots,n)$ in order to evaluate the estimated accuracy.

For simplicity, we define the following relations:

$$\underline{\omega}(i)=\begin{pmatrix} \alpha_i & \beta_i & \gamma_i \end{pmatrix} \quad (27)$$

$$\underline{\delta}(i)=\begin{pmatrix} \underline{\omega}(i) & 1 \end{pmatrix} \quad (28)$$

Then from Eq.(12), we can obtain the following matrix:

$$A(i)=\begin{pmatrix} \underline{\delta}(1) \\ \underline{\delta}(2) \\ \vdots \\ \underline{\delta}(i) \end{pmatrix} \quad (29)$$

4.2. Observability

The following property shows that we can determine whether $A(i)^T A(i)$ and $A(i)^T V(i)^{-1} A(i)$ have inverses using row vectors of $A(i)$.

(Property4) Let $4 \le i$. Then $A(i)^T A(i)$ and $A(i)^T V(i)^{-1} A(i)$ are invertible if and only if we can find 4 independent $\underline{\delta}(j)$ $(j=1,2,\cdots,i)$.

The property4 can be proved using the theory of linear algebra.

Eq.(9) shows that $\underline{\omega}(i)$ is the unit vector from the reference point i to the target location. Therefore, it becomes easy to understand the property4 as shown in the following property when we can determine whether $A(i)^T A(i)$ and $A(i)^T V(i)^{-1} A(i)$ have inverses only using $\underline{\omega}(i)$.

(Property5) Let $4 \le i$. Let k be one of $1,2,\cdots,i$. Then $A(i)^T A(i)$ and $A(i)^T V(i)^{-1} A(i)$ are invertible if and only if we can find 3 independent $\underline{\omega}(j)-\underline{\omega}(k)$ $(j=1,2,\cdots,i,j\neq k)$.

The property5 can be proved using the determinant.

From the property4, if the row rank of $A(i)$ is 4, $A(i)^T V(i)^{-1} A(i)$ is invertible. Therefore, in this case, $\underline{\hat{a}}(i)$ can be computed using Eq.(17). However, even in the divergence case such as the determinant of $A(i)^T V(i)^{-1} A(i)$ is approximately equal to 0, the row rank of $A(i)$ can be four due to the effect of round-off errors and initial estimate errors.

By the way, the minimum singular values of $A(i)$ (that is the square root of the minimum eigenvalue of $A(i)^T A(i)$) is positive if and only if the row rank of $A(i)$ is four. Here, it is well known that the eigenvalue of a matrix is continuous function of elements of the matrix, but the rank of a matrix is not.

On the other hand, the determinant of $A(i)^T A(i)$ is not 0 if and only if $A(i)^T A(i)$ has an inverse. Here, the determinant of a matrix is the product of all its eigenvalues. Therefore, the determinant may be large even when the minimum singular value is approximately equal to 0.

Namely, the minimum singular value of the matrix $A(i)$ can be expected as the better measure for unique solution and the estimated location accuracy.

Therefore, we propose a new measure of estimated target location accuracy using the minimum singular value of $A(i)$.

4.3. Upper and lower bounds for the estimated errors

The following property shows the upper and lower bounds for the estimated error covariance matrix. It also shows that we can compute $\underline{\hat{a}}(i)$ without divergence when the minimum singular value $\lambda_{\min}(i)$ of the matrix $A(i)$ is large.

(Property6) Let $4 \le i$. Assume that we can find 4 independent $\underline{\delta}(j)$ $(j = 1,2,\cdots,i)$. Let $\lambda_{\min}(i)$ be the minimum singular value and $\lambda_{\max}(i)$ be the maximum singular value of the matrix $A(i)$. Let $\sigma^2_{\min}(i)$ be the minimum value and $\sigma^2_{\max}(i)$ be the maximum value of the diagonal elements of the matrix $V(i)$. Then the following result holds:

$$\frac{\sigma^2_{\min}(i)}{\lambda_{\max}(i)^2} I_4 \le \left[A(i)^T V(i)^{-1} A(i)\right]^{-1} \le \frac{\sigma^2_{\max}(i)}{\lambda_{\min}(i)^2} I_4 \qquad (30)$$

The property6 can be proved using the characteristics of the positive definite matrix.

4.4. Effect of the number of the range measurements

The following property shows that both $\lambda_{\min}(i)$ and $\lambda_{\max}(i)$ become large when the number of the range measurements increases.

(Property7) Let $4 \le i$. Then the following results hold:

$$\lambda_{\min}(i) \le \lambda_{\min}(i+1) \qquad (31)$$

$$\lambda_{\max}(i) \le \lambda_{\max}(i+1) \qquad (32)$$

The property7 can be proved using the characteristics of the positive semi-definite matrix.

When the number of the range measurements increases, the property6 and 7 show that a computation of the inverse of $A(i)^T V(i)^{-1} A(i)$ becomes hard to diverge by any target and reference points geometry.

The following property shows that we could obtain better estimation (see Eq.(19)) when the number of the range measurements increases.

(Property8) When

$$0 < A(i)^T V(i)^{-1} A(i) \qquad (33)$$

holds, the following result holds:

$$\left[A(i+1)^T V(i+1)^{-1} A(i+1)\right]^{-1} \le \left[A(i)^T V(i)^{-1} A(i)\right]^{-1} \qquad (34)$$

The property8 can be proved using the characteristics of the positive semi-definite matrix.

5. Consideration

5.1. TOA without the clock offset

When the clock offset does not exist, we can use the following matrix in stead of Eq.(12).

$$A(i)=\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \vdots & \vdots & \vdots \\ \alpha_i & \beta_i & \gamma_i \end{pmatrix} \quad (35)$$

In this case, the property4 becomes easy to understand as the following property.

(Property9) $A(i)^T A(i)$ and $A(i)^T V(i)^{-1} A(i)$ are invertible if and only if we can find 3 independent $\underline{\omega}(j) \quad (j=1,2,\cdots,i)$.

5.2. Relation between DOP and singular values

If $A(i)^T A(i)$ is a diagonal matrix, we can obtain the following equation:

$$A(i)^T A(i) = diag\left\{ \frac{1}{\sigma_x^2}, \frac{1}{\sigma_y^2}, \frac{1}{\sigma_z^2}, \frac{1}{\sigma_t^2} \right\} \quad (36)$$

Since the eigenvalues of a diagonal matrix are equal to its diagonal elements, we can obtain the following equation:

$$\lambda_{\min}(i)^2 = \frac{1}{\max.\left\{ \sigma_x^2, \sigma_y^2, \sigma_z^2, \sigma_t^2 \right\}} \quad (37)$$

$$\lambda_{\max}(i)^2 = \frac{1}{\min.\left\{ \sigma_x^2, \sigma_y^2, \sigma_z^2, \sigma_t^2 \right\}} \quad (38)$$

In general, when $A(i)^T A(i)$ may not be diagonal, we can obtain the following equation:

$$\frac{1}{\lambda_{\max}(i)^2} \le \min.\left\{ \sigma_x^2, \sigma_y^2, \sigma_z^2, \sigma_t^2 \right\} \le \max.\left\{ \sigma_x^2, \sigma_y^2, \sigma_z^2, \sigma_t^2 \right\} \le \frac{1}{\lambda_{\min}(i)^2} \quad (39)$$

The following property shows the relation between a GDOP and singular values. It also shows that GDOP approaches infinity when $\lambda_{\min}(i)$ approaches 0. Here, $\lambda_{\min}(i)$ can be computed even when $A(i)^T A(i)$ is not invertible. But GDOP cannot be computed.

(Property10) Let $\lambda_{\min}(i)$ be positive and $\lambda_j(i) \quad (j=1,2,3,4)$ be singular values of $A(i)$. Then the following equation holds:

$$(GDOP)^2 = \sum_{j=1}^{4} \frac{1}{\lambda_j^2(i)} \quad (40)$$

The property10 can be proved using the characteristics of the trace of a matrix[6].

5.3. Example of TOA in a 2-dimentional space

In this section, for simplicity, we illustrate an example of TOA in a 2-dimentional space without the clock offset.

Let the variance of two range measurements be 1.

In this case, from Eqs.(9) and (27), we can obtain the following unit vectors:

$$\underline{\omega}(j) = \begin{pmatrix} \alpha_j & \beta_j \end{pmatrix} \quad (j=1,2) \quad (41)$$

Let the target location be $(0,0)^T$, reference point1 be $(1,0)^T$ and reference point2 be $(\sin\theta, \cos\theta)^T$ in a x-y plane as shown in Fig.2.

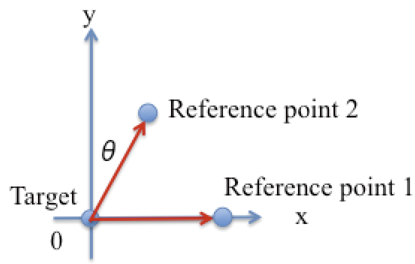

Fig.2 Reference points and a target

Let the initial estimate of the target location be $(0,0)^T$. Then we obtain the following results:

$$\underline{\omega}(1) = \begin{pmatrix} -1 & 0 \end{pmatrix}, \ \underline{\omega}(2) = \begin{pmatrix} -\sin\theta & -\cos\theta \end{pmatrix} \quad (42)$$

We can conclude that $A(2)^T A(2)$ is invertible if and only if $\underline{\omega}(1)$ and $\underline{\omega}(2)$ are linearly independent like the property9.

Let $0 \le \theta \le \pi$. From Eq.(42), we can conclude that $A(2)^T A(2)$ is invertible when $\theta \ne \pi/2$ and $A(2)^T A(2)$ is not invertible when $\theta = \pi/2$

By the way, we can obtain the following matrices from Eq.(42):.

$$A(2) = \begin{pmatrix} -1 & 0 \\ -\sin\theta & -\cos\theta \end{pmatrix} \quad (43)$$

$$A(2)^T A(2)=\begin{pmatrix} 1+\sin^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \cos^2\theta \end{pmatrix} \tag{44}$$

$$\left[A(2)^T A(2)\right]^{-1}=\begin{pmatrix} 1 & -\dfrac{\sin\theta}{\cos\theta} \\ -\dfrac{\sin\theta}{\cos\theta} & \dfrac{1+\sin^2\theta}{\cos^2\theta} \end{pmatrix} \tag{45}$$

From the diagonal elements of Eq.(45), the results are:

$$\sigma_x^2=1,\sigma_y^2=\frac{1+\sin^2\theta}{\cos^2\theta} \tag{46}$$

$$\sigma_x^2+\sigma_y^2=\frac{2}{\cos^2\theta} \tag{47}$$

From Eq.(44), we can obtain the following characteristic polynomial:

$$\left|A(2)^T A(2)-\lambda I_2\right|=\lambda^2-2\lambda+\cos^2\theta \tag{48}$$

Here, $|A|$ indicates the determinant of a matrix A.

From Eq.(48) the results are:

$$\lambda_{\min}^2(2)=1-\sin\theta,\lambda_{\max}^2(2)=1+\sin\theta \quad (0\le\theta\le\pi) \tag{49}$$

$$\frac{1}{\lambda_{\min}^2(2)}+\frac{1}{\lambda_{\max}^2(2)}=\frac{2}{\cos^2\theta} \tag{50}$$

Eqs.(47) and (50) are the same value as shown in the property10.

From Eqs.(46) and (49) , the results are:

$$\sigma_x^2=\sigma_y^2=\frac{1}{\lambda_{\min}^2(2)}=\frac{1}{\lambda_{\max}^2(2)}=1 \quad (\theta=0,\pi) \tag{51}$$

$$\sigma_x^2=1,\sigma_y^2=3 \quad (\theta=\pi/4) \tag{52}$$

$$\frac{1}{\lambda_{\min}^2(2)}=2+\sqrt{2}\approx 3.41,\frac{1}{\lambda_{\max}^2(2)}=2-\sqrt{2}\approx 0.59 \quad (\theta=\pi/4) \tag{53}$$

Eq.(51) shows that there is no difference in estimated errors between DOP and the singular value analysis when $\underline{\omega}(1)$ and $\underline{\omega}(2)$ are perpendicular.

However, estimated errors of DOP differ from those of the singular value analysis when $\theta=\pi/4$. This is because DOPs neglect the rotation of axes.

By the way, Eq.(47) shows that estimated errors become very large when θ is approximately $\pi/2$ ($\underline{\omega}(1)$ and $\underline{\omega}(2)$ have a small angle). In this case, $\lambda_{\min}(2)$ is approximately 0. Fig.3 illustrates the relation between the singular values and DOP as a function of θ.

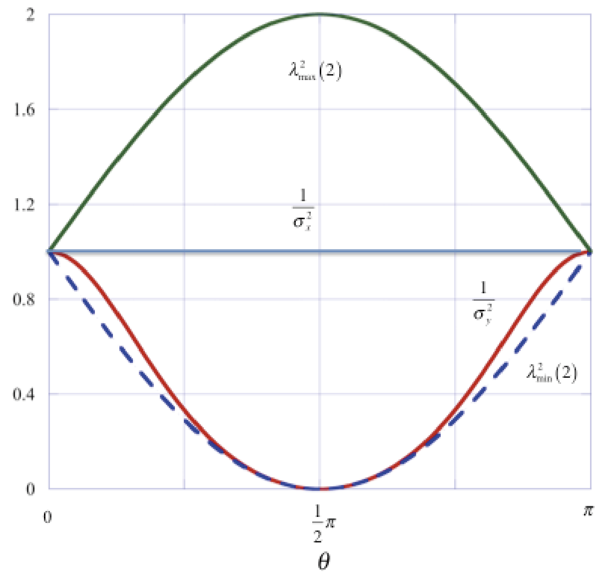

Fig.3 Relation between singular values and an DOP

6. Conclusions

In this paper, we have provided the measure of estimated target location accuracy for a TOA system. This measure is the minimum singular value of the placement matrix. We have shown that we could obtain good estimation when this minimum singular value is large. We have illustrated that we can estimate the variance of estimated error using this minimum singular value and the variances of a range measurement error. We also illustrate that this minimum singular value becomes large and the covariance matrix of estimated location errors becomes small when the number of range measurements increases.

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